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DEEPAK PRAKASHAN

Applied Mathematics

SECOND

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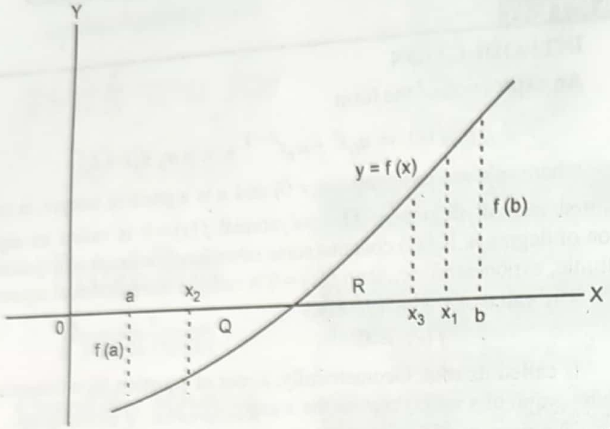
between x_1 and x_2. Then the third approximation to the root is $x_3 = \dfrac{x_1 + x_2}{2}$ and so on.



This method is simple but slowly convergent. We have to repeated above procedure till two successive values coincide upto 3 places of decimal.

ILLUSTRATIVE EXAMPLES

Example 1.1. Find a root of the equation $f(x) = 8x^3 - 2x - 1 = 0$, *using the bisection method.*

Sol. Given,

$$f(x) = 8x^3 - 2x - 1$$
$$f(0) = 0 - 0 - 1 = -1 \text{ (–ve)}$$
$$f(1) = 8 - 2 - 1 = 5 \text{ (+ve)}$$

Since $f(0)$ is –ve and $f(1)$ is +ve, a root lies between 0 and 1

First Approximation

Let

$$a = 0, b = 1$$
$$x_1 = \frac{a+b}{2} = \frac{0+1}{2} = 0{\cdot}5$$

$\therefore$

$$f(x_1) = f(0{\cdot}5)$$
$$= 8(0{\cdot}5)^3 - 2(0{\cdot}5) - 1$$
$$= 1 - 1 - 1 = -1 < 0 \text{ (–ve)}$$

⇒ The root lies between 0·5 and 1

Second Approximation

$$x_2 = \frac{b + x_1}{2} = \frac{1 + 0{\cdot}5}{2} = 0{\cdot}75$$
$$f(x_2) = f(0{\cdot}75)$$

$$= 8(0{\cdot}75)^3 - 2(0{\cdot}75) - 1$$
$$= 3{\cdot}375 - 2{\cdot}5 = 0{\cdot}875 > 0 \text{ (+ve)}$$

⇒ The root lies between 0·5 and 0·75

Third Approximation

$$x_3 = \frac{x_1 + x_2}{2} = \frac{0{\cdot}5 + 0{\cdot}75}{2} = 0{\cdot}625$$
$$f(x_3) = f(0{\cdot}625)$$
$$= 8(0{\cdot}625)^3 - 2(0{\cdot}625) - 1$$
$$= 1{\cdot}953 - 2{\cdot}25 = -0{\cdot}297 < 0 \text{ (–ve)}$$

⇒ The root lies between 0·625 and 0·75

Fourth Approximation

$$x_4 = \frac{x_2 + x_3}{2} = \frac{0{\cdot}75 + 0{\cdot}625}{2} = 0{\cdot}688$$
$$f(x_4) = f(0{\cdot}688)$$
$$= 8(0{\cdot}688)^3 - 2(0{\cdot}688) - 1$$
$$= 2{\cdot}6053 - 2{\cdot}376 = 0{\cdot}2293 > 0 \text{ (+ve)}$$

⇒ The root lies between 0·625 and 0·688

Fifth Approximation

$$x_5 = \frac{x_3 + x_4}{2} = \frac{0{\cdot}625 + 0{\cdot}688}{2} = 0{\cdot}657$$
$$f(x_5) = f(0{\cdot}657)$$
$$= 8(0{\cdot}657)^3 - 2(0{\cdot}657) - 1$$
$$= 2{\cdot}2687 - 2{\cdot}314 = -0{\cdot}0453 < 0 \text{ (–ve)}$$

⇒ The root lies between 0·657 and 0·688

Sixth Approximation

$$x_6 = \frac{x_4 + x_5}{2} = \frac{0{\cdot}688 + 0{\cdot}657}{2} = 0{\cdot}673$$
$$f(x_6) = f(0{\cdot}673)$$
$$= 8(0{\cdot}673)^3 - 2(0{\cdot}673) - 1$$
$$= 2{\cdot}4386 - 2{\cdot}346 = 0{\cdot}0926 > 0 \text{ (+ve)}$$

⇒ The root lies between 0·673 and 0·657

Seventh Approximation

$$x_7 = \frac{x_5 + x_6}{2} = \frac{0{\cdot}657 + 0{\cdot}673}{2} = 0{\cdot}665$$
$$f(x_7) = f(0{\cdot}665)$$
$$= 8(0{\cdot}665)^3 - 2(0{\cdot}665) - 1$$
$$= 2{\cdot}3526 - 1{\cdot}33 - 1 = 0{\cdot}0226 > 0 \text{ (+ve)}$$

⇒ The root lies between 0·665 and 0·657

Eighth Approximation

$$x_8 = \frac{x_5 + x_7}{2} = \frac{0{\cdot}657 + 0{\cdot}665}{2} = 0{\cdot}661$$

$$\begin{aligned} f(x_8) &= f(0{\cdot}661) \\ &= 8\,(0{\cdot}661)^3 - 2\,(0{\cdot}661) - 1 \\ &= 2{\cdot}3104 - 1{\cdot}322 - 1 \\ &= 2{\cdot}3104 - 2{\cdot}322 = -\,0{\cdot}0116 < 0\,(-\text{ve}) \end{aligned}$$

$\Rightarrow$ The root lies between 0·665 and 0·661

Ninth Approximation

$$x_9 = \frac{x_7 + x_8}{2} = \frac{0{\cdot}665 + 0{\cdot}661}{2} = 0{\cdot}663$$

$$\begin{aligned} f(x_9) &= f(0{\cdot}663) \\ &= 8\,(0{\cdot}663)^3 - 2\,(0{\cdot}663) - 1 \\ &= 2{\cdot}3315 - 2{\cdot}326 = 0{\cdot}0055\ (+\text{ve}) \end{aligned}$$

$\Rightarrow$ The root lies between 0·663 and 0·661

Tenth Approximation

$$x_{10} = \frac{x_8 + x_9}{2} = \frac{0{\cdot}661 + 0{\cdot}663}{2} = 0{\cdot}662$$

$$\begin{aligned} f(x_{10}) &= f(0{\cdot}662) \\ &= 8\,(0{\cdot}662)^3 - 2\,(0{\cdot}662) - 1 \\ &= 2{\cdot}32094 - 1{\cdot}324 - 1 = -\,0{\cdot}00306\ (-\text{ve}) \end{aligned}$$

Therefore, from x_9 and x_{10}, the root is 0·66

Thus, the approximate root is 0·66 **Ans.**

Example 1.2. Find a root of the equation $x^3 - 4x - 9 = 0$, using bisection method in seven stages.

Sol. Let

$$\begin{aligned} f(x) &= x^3 - 4x - 9 \\ f(1) &= 1 - 4 - 9 = -\,12\ (-\text{ve}) \\ f(2) &= 8 - 8 - 9 = -\,9\ (-\text{ve}) \\ f(3) &= 27 - 12 - 9 = 6\ (+\text{ve}) \end{aligned}$$

Since $f(2)$ is –ve and $f(3)$ is +ve, a root lies between 2 and 3.

First Approximation

Let,

$$a = 2,\ b = 3$$

$$x_1 = \frac{2+3}{2} = 2{\cdot}5$$

$$\begin{aligned} f(x_1) &= f(2{\cdot}5) \\ &= (2{\cdot}5)^3 - 4(2{\cdot}5) - 9 \\ &= 15{\cdot}625 - 10 - 9 = -\,3{\cdot}375\ (-\text{ve}) \end{aligned}$$

$\Rightarrow$ Therefore, the root lies between 2·5 and 3

Second Approximation

$$x_2 = \frac{2{\cdot}5 + 3}{2} = 2{\cdot}75$$

$$\begin{aligned} f(x_2) &= f(2{\cdot}75) \\ &= (2{\cdot}75)^3 - 4(2{\cdot}75) - 9 \\ &= 20{\cdot}7969 - 11 - 9 = 0{\cdot}7969\ (+\text{ve}) \end{aligned}$$

$\Rightarrow$ Therefore, the root lies between 2·5 and 2·75

Third Approximation

$$x_3 = \frac{2{\cdot}5 + 2{\cdot}75}{2} = 2{\cdot}625$$

$$\begin{aligned} f(x_3) &= f(2{\cdot}625) \\ &= (2{\cdot}625)^3 - 4(2{\cdot}625) - 9 \\ &= 18{\cdot}0879 - 10{\cdot}5 - 9 = -\,1{\cdot}4121\ (-\text{ve}) \end{aligned}$$

$\Rightarrow$ Therefore, the root lies between 2·75 and 2·625

Fourth Approximation

$$x_4 = \frac{2{\cdot}75 + 2{\cdot}625}{2} = 2{\cdot}6875$$

$$\begin{aligned} f(x_4) &= f(2{\cdot}6875) \\ &= (2{\cdot}6875)^3 - 4(2{\cdot}6875) - 9 \\ &= 19{\cdot}4109 - 10{\cdot}75 - 9 = -\,0{\cdot}3391\ (-\text{ve}) \end{aligned}$$

$\Rightarrow$ Therefore, the root lies between 2·75 and 2·6875

Fifth Approximation

$$x_5 = \frac{2{\cdot}75 + 2{\cdot}6875}{2} = 2{\cdot}71875$$

$$\begin{aligned} f(x_5) &= f(2{\cdot}71875) \\ &= (2{\cdot}71875)^3 - 4(2{\cdot}71875) - 9 \\ &= 20{\cdot}0959 - 10{\cdot}875 - 9 = 0{\cdot}2209\ (+\text{ve}) \end{aligned}$$

$\Rightarrow$ Therefore, the root lies between 2·6875 and 2·71875

Sixth Approximation

$$x_6 = \frac{2{\cdot}6875 + 2{\cdot}71875}{2} = 2{\cdot}7031$$

$$\begin{aligned} f(x_6) &= f(2{\cdot}7031) \\ &= (2{\cdot}7031)^3 - 4(2{\cdot}7031) - 9 \\ &= 19{\cdot}7509 - 10{\cdot}8124 - 9 = -\,0{\cdot}0615\ (-\text{ve}) \end{aligned}$$

$\Rightarrow$ Therefore, the root lies between 2·71875 and 2·7031

Seventh Approximation

$$x_7 = \frac{2{\cdot}71875 + 2{\cdot}7031}{2} = 2{\cdot}7109$$

Therefore, the root lies between 2·7031 and 2·7109

Hence, the root is 2·70 approximately. **Ans.**

Example 1.3. Find the real root of the equation $xe^x - 3 = 0$ by bisection method, correct to three places of decimal.

Sol. Let $f(x) = xe^x - 3$

Now $f(1) = e - 3 = -0.28172$ (–ve)

and $f(1.5) = 1.5e^{1.5} - 3 = 3.72253$ (+ve)

$f(1)$ and $f(1.5)$ are of opposite signs, so at least one root of the given equation lies between 1 and 1.5.

First Approximation

Let $a = 1, b = 1.5$

Then
$$x_1 = \frac{1+1.5}{2} = 1.25$$
$$f(x_1) = f(1.25)$$
$$= (1.25)\,e^{1.25} - 3 = 1.3629 \text{ (+ve)}$$

∴ The required root lies between 1 and 1.25

Second Approximation
$$x_2 = \frac{1+1.25}{2} = 1.125$$
$$f(x_2) = f(1.125)$$
$$= (1.125)\,e^{1.125} - 3 = 0.4652 \text{ (+ve)}$$

∴ The required root lies between 1 and 1.125

Third Approximation
$$x_3 = \frac{1+1.125}{2} = 1.0625$$
$$f(x_3) = f(1.0625)$$
$$= (1.0625)\,e^{1.0625} - 3 = 0.0744 \text{ (+ve)}$$

∴ The required root lies between 1 and 1.0625

Fourth Approximation
$$x_4 = \frac{1+1.0625}{2} = 1.03125$$
$$f(x_4) = f(1.03125)$$
$$= (1.03125)\,e^{1.03125} - 3 = -0.10779 \text{ (–ve)}$$

∴ The required root lies between 1.0625 and 1.03125

Fifth Approximation
$$x_5 = \frac{1.0625+1.03125}{2} = 1.0469$$
$$f(x_5) = f(1.0469)$$
$$= (1.0469)\,e^{1.0469} - 3 = -0.01758 \text{ (–ve)}$$

∴ The required root lies between 1.0625 and 1.0469

Sixth Approximation
$$x_6 = \frac{1.0625+1.0469}{2} = 1.0547$$
$$f(x_6) = f(1.0547)$$
$$= (1.0547)\,e^{1.0547} - 3 = 0.02816 \text{ (+ve)}$$

∴ The required root lies between 1.0469 and 1.0547

Seventh Approximation
$$x_7 = \frac{1.0469+1.0547}{2} = 1.0508$$
$$f(x_7) = f(1.0508)$$
$$= (1.0508)\,e^{1.0508} - 3 = 0.00522 \text{ (+ve)}$$

∴ The required root lies between 1.0469 and 1.0508

Eighth Approximation
$$x_8 = \frac{1.0469+1.0508}{2} = 1.04885$$
$$f(x_8) = f(1.04885)$$
$$= (1.04885)\,e^{1.04885} - 3 = -0.006197 \text{ (–ve)}$$

∴ The required root lies between 1.0508 and 1.04885

Ninth Approximation
$$x_9 = \frac{1.0508+1.04885}{2} = 1.049825$$
$$f(x_9) = f(1.049825)$$
$$= (1.049825)\,e^{1.049825} - 3$$
$$= -0.000491 \text{ (–ve)}$$

Therefore, from x_8 and x_9 we can see that $f(x_8)$ and $f(x_9)$ are nearly equal to zero.

Hence, the root correct to three decimal places is 1.049 **Ans.**

6 *Example 1.4. Find cube root of 26, correct to three places of decimal by bisection method.*

Sol. Let $x = (26)^{1/3}$

$\Rightarrow$ $x^3 - 26 = 0$

$\Rightarrow$ $f(x) = x^3 - 26$

∴ $f(1) = 1 - 26 = -25$

$f(2) = 8 - 26 = -18$ (–ve)

$f(3) = 27 - 26 = 1$ (+ve)

∴ Root lies between 2 and 3

First Approximation

Let, $a = 2, b = 3$

$$[E^n - {}^nc_1E^{n-1} + {}^nc_2E^{n-2} - {}^nc_3E^{n-3} + \dots + (-1)^n\, {}^nc_nE^0]\, y_1 = 0$$

$$E^ny_1 - {}^nc_1E^{n-1}y_1 + {}^nc_2E^{n-2}y_1 - {}^nc_3E^{n-3}y_1 + \dots + (-1)^n\, {}^nc_n\, E^0y_1 = 0$$

$$y_{n+1} - {}^nc_1y_n + {}^nc_2y_{n-1} - {}^nc_3y_{n-2} + \dots + (-1)^ny_1 = 0 \qquad \text{...(iv)}$$

i.e., equation (iii) and (iv) are two required equations for two missing values. On solving these two equations we get required values.

ILLUSTRATIVE EXAMPLES

Example 2.11. *Using Newton's forward interpolation formula, find the value of y for x = 5 :*

x :	4	6	8	10
y :	1	3	8	16

Sol. The forward differences are calculated and tabulated as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
			3	
6	3			0
		5		
			3	
8	8			
		8		
10	16			

Here, $x = 5, x_0 = 4, y_0 = 1, h = 2$

Since $p = \dfrac{x - x_0}{h}$

$$p = \frac{5-4}{2} = 0.5$$

Now by the Newton forward interpolation formula

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)\,(p-2)}{3!}\Delta^3 y_0 + \dots$$

$$= 1 + 2(0.5) + \frac{0.5(0.5-1)}{2!} \times 3 + \frac{0.5(0.5-1)\,(0.5-2)}{3!} \times 0$$

$$= 1 + 1 - 0.375 + 0$$

$$y_p = 1.625$$

Ans.

Example 2.12. *Using Newton's interpolation formula, find the value of f (1.85), if*

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
y :	5.474	6.050	6.686	7.389	8.166	9.205	9.974

Sol. The difference table is follows :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.7	5.474						
		0.576					
1.8	6.050		0.06				
		0.636		0.007			
1.9	6.686		0.067		0		
		0.703		0.007		0.181	
2.0	7.389		0.074		0.181		− 1.082
		0.777		0.188		− 0.901	
2.1	8.166		0.262		− 0.72		
		1.039		− 0.532			
2.2	9.205		− 0.27				
		0.769					
2.3	9.974						

Given, $x = 1.85, x_0 = 1.7, h = 0.1$

$$\therefore \quad p = \frac{x - x_0}{h} = \frac{1.85 - 1.7}{0.1} = 1.5$$

Now by the Newton's forward interpolation formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)\,(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)\,(p-2)\,(p-3)}{4!}\Delta^4 y_0 + \dots$$

$$= 5.474 + 1.5 \times 0.576 + \frac{1.5 \times 0.5 \times 0.06}{2!} + \frac{1.5 \times 0.5 \times (-0.5) \times 0.007}{3!} + 0$$

$$+ \frac{1.5 \times 0.5 \times (-0.5) \times (-1.5) \times (-2.5) \times 0.181}{5!}$$

$$+ \frac{1.5 \times 0.5 \times (-0.5) \times (-1.5) \times (-2.5) \times (-3.5) \times (-1.082)}{6!}$$

$$= 5.474 + 0.864 + 0.0225 - 0.0004375 - 0.002121 - 0.007396$$

Ans.

$$y = f(1.85) = 6.350546$$

Example 2.13. *The population of a town in the decennial census was as given below. Estimate the population for the year 1895.*

Year x	:	1891	1901	1911	1921	1931
Population y (In thousands)	:	46	66	81	93	101

Given, $x = 84, x_n = 90, y_n = 304, h = 10$

$$p = \frac{x - x_n}{h} = \frac{84 - 90}{10} = -0{\cdot}6$$

Since $x = 84$ is near the end of the table, we use Newton's backward interpolation formula,

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$= 304 + (-0{\cdot}6)(28) + \frac{(-0{\cdot}6)(-0{\cdot}6+1)}{2} \times 2 + 0$$

$$= 304 - 16{\cdot}8 - 0{\cdot}24$$

$$f(84) = \mathbf{286{\cdot}96}$$ **Ans.**

Example 2.19. *The population of a country in the decennial census were as under. Estimate the population for the year 1975.*

Year x	:	*1941*	*1951*	*1961*	*1971*	*1981*
Population y	:	*46*	*67*	*83*	*95*	*102*
(In Lakhs)						

Sol. The difference table is :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1941	46				
		21			
1951	67		−5		
		16		1	
1961	83		−4		−2
		12		−1	
1971	95		−5		
		7			
1981	102				

Given, $x = 1975, x_n = 1981, y_n = 102, h = 10$

$$p = \frac{x - x_n}{h} = \frac{1975 - 1981}{10} = -0{\cdot}6$$

By Newton's backward interpolation formula

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$$

$$= 102 + (-0{\cdot}6) \times 7 + \frac{(-0{\cdot}6)(-0{\cdot}6+1)}{2} \times (-5)$$

$$+ \frac{(-0{\cdot}6)(-0{\cdot}6+1)(-0{\cdot}6+2)}{6} \times (-1)$$

$$+ \frac{(-0{\cdot}6)(-0{\cdot}6+1)(-0{\cdot}6+2)(-0{\cdot}6+3)}{24} \times (-2)$$

$$= 102 - 4{\cdot}2 + 0{\cdot}6 + 0{\cdot}056 + 0{\cdot}0672$$

$$= \mathbf{98{\cdot}5232}$$ **Ans.**

Example 2.20. *Apply Newton's backward interpolation formula to the data below, to obtain a polynomial of degree four in x and evaluate f (5).*

x :	*1*	*2*	*3*	*4*	*5*
y :	*1*	*−1*	*1*	*−1*	*1*

Hence, evaluate y for x = 5.

Sol. The difference table is,

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
		−2			
2	−1		4		
		2		−8	
3	1		−4		16
		−2		8	
4	−1		4		
		2			
5	1				

Given, $x_n = 5, y_n = 1, h = 1$

$$p = \frac{x - x_n}{h} = \frac{x-5}{1} = x - 5$$

By Newton's backward interpolation formula,

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$+ \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$= 1 + (x-5)(2) + \frac{(x-5)(x-5+1)}{2!}(4)$$

$$+ \frac{(x-5)(x-5+1)(x-5+2)}{3!}(8)$$

$$+ \frac{(x-5)(x-5+1)(x-5+2)(x-5+3)}{4!}(16)$$

$$= 1 + 2x - 10 + 2(x-5)(x-4) + \frac{4}{3}(x-5)(x-4)(x-3)$$

$$+ \frac{2}{3}(x-5)(x-4)(x-3)(x-2)$$

$$= 1 + 2x - 10 + 2x^2 - 18x + 40 + \frac{4}{3}(x^3 - 12x^2 + 47x - 60)$$

$$+ \frac{2}{3}(x^4 - 14x^3 + 71x^2 - 154x + 120)$$

$$y = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31$$ **Ans.**

The polynomial is $y = f(x) = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31$

At $x = 5$,
$$f(5) = \frac{2}{3}(5)^4 - 8(5)^3 + \frac{100}{3}(5)^2 - 56 \times 5 + 31$$
$$= 1$$ **Ans.**

Example 2.21. *Apply Stirling's formula to compute $y_{12.2}$ from the following data :*

x :	10	11	12	13	14
y :	0.23967	0.28060	0.31788	0.35209	0.38368

Sol. Taking the origin at $x_0 = 12$, $h = 1$ and $p = \frac{x - x_0}{h}$

Now the difference table is given below :

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	−2	0.23967				
			0.04093			
11	−1	0.28060		−0.00365		
			0.03728		**0.00058**	
12	**0**	**0.31788**		**−0.00307**		**−0.00013**
			0.03421		**0.00045**	
13	1	0.35209		−0.00262		
			0.03159			
14	2	0.38368				

At $x = 12.2$, $p = \frac{x - x_0}{h} = \frac{12.2 - 12}{1} = 0.2$

By the Stirling's formula, we know that

$$y = y_0 + p\left\{\frac{\Delta y_{-1} + \Delta y_0}{2}\right\} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots$$

$$= 0.31788 + 0.2\left\{\frac{0.03728 + 0.03421}{2}\right\} + \frac{(0.2)^2}{2}(-0.00307) + \frac{(0.2)\{(0.2)^2 - 1\}}{6}\left\{\frac{0.00058 + 0.00045}{2}\right\} + \frac{(0.2)^2\{(0.2)^2 - 1\}}{24}(-0.00013)$$

$$= 0.31788 + 0.007149 - 0.0000614 - 0.000016 + 0.0000002$$
$$= \mathbf{0.324952}$$ **Ans.**

Example 2.22. *From the following table, find the value of log 337.5 by Stirling formula :*

x	:	310	320	330
$\log_{10} x$	:	2.4913617	2.5051500	2.5185139
x	:	340	350	360
$\log_{10} x$	:	2.5314789	2.5440680	2.5563025

Sol. Taking 330 as the origin and 10 as the unit.

$$x_0 = 330,\ x = 337.5,\ h = 10$$
$$p = \frac{337.5 - 330}{10} = 0.75 = \frac{3}{4}$$

The difference table is as under :

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
310	−2	2.4913617					
			0.0137883				
320	−1	2.5051500		−0.0004244			
			0.0133639		**0.0000255**		
330	**0**	**2.5185139**		**−0.0003989**		**-25×10^{-7}**	
			0.0129650		**0.000023**		8×10^{-7}
340	1	2.5314789		−0.0003759		-17×10^{-7}	
			0.0125891		0.0000213		
350	2	2.5440680		−0.0003546			
			0.0122345				
360	3	2.5563025					

By Stirling formula,

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots$$

CHAPTER 3

NUMERICAL DIFFERENTIATION AND INTEGRATION

3.1 NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i). To compute $\frac{dy}{dx}$, we first replace the exact relation $y = f(x)$ by the best interpolating polynomial $y = \phi(x)$ and then differentiate the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of x at which $\frac{dy}{dx}$ is desired.

If the values of x are equi-spaced and $\frac{dy}{dx}$ is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For value near the middle of the table, $\frac{dy}{dx}$ is calculated by means of Stirling's formula.

If the values of x are not equi-spaced, we use Newton's divided difference formula to represent the function.

Hence corresponding to each of the interpolation formulae, we can derive a formula for finding the derivative.

3.1.1 Derivatives Using Newton's Forward Difference Formula :

We know that by Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \quad \ldots\text{(i)}$$

Differentiating both sides of equation (i) with respect to p, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots \quad \ldots\text{(ii)}$$

Since $p = \frac{x - x_0}{h}$, therefore $\frac{dp}{dx} = \frac{1}{h}$

Now $\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{dy}{dp}\cdot\frac{1}{h}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \ldots\right] \quad \ldots\text{(iii)}$$

At $x = x_0$, $p = 0$, hence putting $p = 0$ in equation (iii), we get

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5} - \frac{1}{6}\Delta^6 y_0 + \ldots\right] \quad \ldots\text{(iv)}$$

Differentiating equation (iii) again with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

$$= \frac{d}{dp}\left(\frac{dy}{dx}\right)\cdot\frac{1}{h}$$

$$= \frac{1}{h^2}\frac{d}{dp}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \ldots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \ldots\right] \quad \ldots\text{(v)}$$

Putting $p = 0$ in equation (v), we get

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 \ldots\right] \quad \ldots\text{(vi)}$$

Similarly,

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \frac{7}{4}\Delta^5 y_0 + \ldots\right] \quad \ldots\text{(vii)}$$

and so on.

Alternative Method : We know that

$$e^{hD} = E = 1 + \Delta$$

$$hD = \log(1+\Delta)$$

$$hD = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots$$

$$\therefore \quad D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right] \quad \ldots\text{(i)}$$

Multiply by y both sides of equation (i), we get

$$Dy = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y - \frac{\Delta^2 y}{2} + \frac{\Delta^3 y}{3} - \frac{\Delta^4 y}{4} + \ldots\right]$$

At $x = x_0$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

Now by equation (i),

$$D^2 = \frac{1}{h^2}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right]^2$$

$$D^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \ldots\right] \quad \ldots\text{(ii)}$$

Multiply by y both sides of equation (ii), we get

$$D^2 y = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y - \Delta^3 y + \frac{11}{12}\Delta^4 y - \frac{5}{6}\Delta^5 y + \ldots\right]$$

At $x = x_0$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\right]$$

Similarly,

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \frac{7}{4}\Delta^5 y_0 - \ldots\right]$$

3.1.2 Derivatives Using Newton's Backward Difference Formula : We know that by Newton's backward interpolation formula is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots \quad \ldots\text{(i)}$$

where $p = \frac{x - x_n}{h}$

Differentiating both sides of equation (i) with respect to p, we have

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2!} + \nabla^2 y_n + \frac{3p^2 + 6p + 2}{3!}\nabla^3 y_n + \ldots$$

Since $p = \frac{x - x_0}{h}$, therefore $\frac{dp}{dx} = \frac{1}{h}$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{dy}{dp}\cdot\frac{1}{h} \qquad \left[\because \frac{dp}{dx} = \frac{1}{h}\right]$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2 + 6p + 2}{3!}\nabla^3 y_n + \ldots\right] \quad \ldots\text{(ii)}$$

At $x = x_n$, $p = 0$, hence putting $p = 0$ in equation (ii), we get

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \ldots\right] \quad \ldots\text{(iii)}$$

Again, differentiating equation (ii) with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{1}{h}\cdot\frac{d}{dp}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2 + 18p + 11}{12}\nabla^4 y_n + \ldots\right] \quad \ldots\text{(iv)}$$

Putting $p = 0$, we get

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right] \quad \ldots\text{(v)}$$

Similarly,

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right] \quad \ldots\text{(vi)}$$

Alternative Method : We know that, $e^{-hD} = 1 - \nabla = E^{-1}$

$$-hD = \log(1 - \nabla)$$

$$-hD = -\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \ldots\right]$$

$$D = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \ldots\right] \quad \ldots\text{(i)}$$

Squaring both sides of equation (i), we get

$$D^2 = \frac{1}{h^2}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \ldots\right]^2$$

$$D^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \ldots\right] \quad \ldots\text{(ii)}$$

Multiply by y both sides of equation (ii), we get

$$D^2 y = \frac{1}{h^2}\left[\nabla^2 y + \nabla^3 y + \frac{11}{12}\nabla^4 y + \frac{5}{6}\nabla^5 y + \ldots\right]$$

At $x = x_n$, $p = 0$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y + \nabla^3 y + \frac{11}{12}\nabla^4 y + \frac{5}{6}\nabla^5 y + \ldots\right]$$

Similarly, $\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \frac{7}{4}\nabla^5 + \ldots\right]$

3.1.3 Derivatives Using Stirling's Formula : By the stirling formula, we know that Stirling's formula is

$$y = y_0 + \frac{p}{1!}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots \quad \text{...(i)}$$

where $p = \frac{x - x_0}{h}$

Now $\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{dy}{dp}\cdot\frac{1}{h}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{h}\left[\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2}\right\} + p\Delta^2 y_{-1} + \frac{(3p^2-1)}{6}\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\} + \frac{(2p^3-p)}{12}\Delta^4 y_{-2} + \ldots\right] \quad \text{...(ii)}$$

At $x = x_0$, $p = 0$. Hence, putting $p = 0$, we get

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2}\right\} - \frac{1}{6}\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\} + \frac{1}{30}\left\{\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right\} + \ldots\right] \quad \text{...(iii)}$$

Differentiating equation (ii), with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + p\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\} + \frac{6p^2-1}{12}\Delta^4 y_{-2} + \ldots\right]$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \ldots\right] \quad \text{...(iv)}$$

Similarly,

$$\left[\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}\left[\frac{1}{2}\{\Delta^3 y_{-1} + \Delta^3 y_{-2}\} + \ldots\right]$$

ILLUSTRATIVE EXAMPLES

Example 3.1. *Find the first and second derivatives of $f(x)$ at $x = 1.5$ if :*

x :	1·5	2·0	2·5	3·0	3·5	4·0
y :	3·375	7·000	13·625	24·000	38·875	59·000

Sol. We have to find the derivative at the point $x = 1.5$ which is at the beginning of the given data. Therefore, we use here the derivatives of Newton's forward interpolation formula. The forward difference table is as follows :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1·5	3·375				
		3·625			
2·0	7·000		3		
		6·625		0·75	
2·5	13·625		3·75		0
		10·375		0·75	
3·0	24·000		4·5		0
		14·875		0·75	
3·5	38·875		5·25		
		20·125			
4·0	59·000				

Here, $x_0 = 1.5$, $y_0 = 3.375$, $\Delta y_0 = 3.625$, $\Delta^2 y_0 = 3$, $\Delta^3 y_0 = 0.75$, $\Delta^4 y_0 = 0$ and $h = 0.5$, $x = 1.5$

$$\because \quad p = \frac{x - x_0}{h} = \frac{1.5 - 1.5}{0.5} = 0$$

$\therefore$ We know that by the first order derivative formula

$$\left(\frac{dy}{dx}\right)_{x_0} = f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \ldots\right]$$

$$\left(\frac{dy}{dx}\right)_{1.5} = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3) + \frac{1}{3}(0.75) - 0\right]$$

$$\left(\frac{dy}{dx}\right)_{1.5} = f'(1.5) = 4.75 \qquad \textbf{Ans.}$$

We know that by the second order derivative formula

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\right]$$

$$= \frac{1}{(0.5)^2}[3 - 0.75 + 0]$$

CHAPTER 4

DIFFERENCE EQUATIONS

4.1 INTRODUCTION

Difference calculus also forms the basis of difference equations which arise in the theory of probability, in the study of electrical networks, in statistical problems and indeed in all situations in which sequential relation exists at various discrete values of the independent variable. Just as the subject of differential equations grew out of differential calculus to become one of the most powerful instruments in the hands of a practical mathematician when dealing with continuous processes in nature, so the subject of difference equations is forcing its way to the fore for the treatment of discontinuous processes.

4.2 DEFINITION OF DIFFERENCE EQUATION

A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Thus
$$\Delta y_{n+1} + 2y_n = 2 \quad \text{...(i)}$$
and
$$\Delta y_{n+1} + \Delta^2 y_{n-1} = 1 \quad \text{...(ii)}$$
are difference equations.

An alternative way of writing a difference equation is as under :

Since,
$$\Delta y_{n+1} = y_{n+2} - y_{n+1}$$

∴ Equation (i) may be written as
$$y_{n+2} - y_{n+1} + 2y_n = 2 \quad \text{...(iii)}$$

Also since,
$$\Delta^2 y_{n-1} = y_{n+1} - 2y_n + y_{n-1}$$

∴ Equation (ii) takes the form
$$y_{n+2} - 2y_n + y_{n-1} = 1 \quad \text{...(iv)}$$

Quite often, difference equations are met under the name of recurrence relations.

4.2.1 Order of Difference Equation

Order of Difference Equation : Order of a difference equation is the difference between the largest and the smallest arguments occurring in the difference equation divided by the unit of increment.

Thus equation (iii) above is of the second order, for,
$$\frac{\text{Largest argument} - \text{Smallest argument}}{\text{Unit of increment}} = \frac{(n+2)-n}{1} = 2$$
and equation (iv) is of the third order, for
$$\frac{(n+2)-(n-1)}{1} = 3$$

Note : While finding the order of a difference equation, it must always be expressed in a [illegible], for the highest power of Δ does not give order of the difference equation.

4.2.2 Solution of Difference Equation

Solution of Difference Equation : Solution of a difference equation is an expression for y_n which satisfies the given difference equation.

The general solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

A particular solution (or particular integral) is that solution which is obtained from the general solution by giving particular values to the constants.

4.3 LINEAR DIFFERENCE EQUATIONS

A linear difference equation is that in which y_{n+1}, y_{n+2} etc., occur to the first degree only and are not multiplied together.

A linear difference equation with constant coefficients is of the form
$$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_r y_n = f(n) \quad \text{...(i)}$$
where, $a_1, a_2 \dots a_r$ are constants.

Now we shall deal with linear difference equations with constant coefficients only. Their properties are analogous to those of linear differential equations with constant coefficients.

4.3.1 Elementary Properties of Linear Difference Equations

Elementary Properties of Linear Difference Equations : If $u_1(n), u_2(n), \dots, u_r(n)$ be r independent solutions of the equation
$$y_{n+r} + a_1 y_{n+r-1} + \dots + a_r y_n = 0 \quad \text{...(ii)}$$
then its complete solution is
$$u_n = c_1 u_1(n) + c_2 u_2(n) + \dots + c_r u_r(n)$$
where, $c_1, c_2, \dots, c_r$ are arbitrary constants.

If v_n is a particular solution of equation (i) then the complete solution of equation (i) is
$$y_n = u_n + v_n$$

The part u_n is called the complementary function (C.F.) and the part v_n is called the particular integral (P.I.) of equation (i). Thus the complete solution of equation (i) is
$$y_n = \text{C.F.} + \text{P.I.}$$

4.4 COMPLEMENTARY FUNCTION

Rules to solve a linear difference equation with constant coefficients having right hand side zero.

(i) To begin with, consider the first order linear equation
$$y_{n+1} - \lambda y_n = 0, \text{ where } \lambda \text{ is a constant.}$$

Rewriting it as
$$\frac{y_{n+1}}{\lambda^{n+1}} - \frac{y_n}{\lambda^n} = 0$$

We have
$$\Delta\left(\frac{y_n}{\lambda_n}\right) = 0$$

Which gives $\dfrac{y_n}{\lambda_n} = c$, a constant.

Thus the solution of $(E-\lambda)y_n = 0$

$$y_n = c\,.\,\lambda^n$$

(ii) Now consider the second order linear equation

$$y_{n+2} + ay_{n+1} + by_n = 0$$

which is symbolic form is

$$(E^2 + aE + b)y_n = 0 \qquad \text{...(i)}$$

Its symbolic coefficient equated to zero

$$E^2 + aE + b = 0$$

is called the auxiliary equation.

Let its root be λ_1, λ_2

Case I. If these roots are real and distinct, then equation (i) is equivalent to

$$(E-\lambda_1)(E-\lambda_2)\,y_n = 0 \qquad \text{...(ii)}$$

or $\quad (E-\lambda_2)(E-\lambda_1)\,y_n = 0 \qquad \text{...(iii)}$

If y_n satisfies the subsidiary equation $(E-\lambda_1)\,y_n = 0$, then it will also satisfy equation (iii).

Similarly, if y_n satisfies the subsidiary equation $(E-\lambda_2)\,y_n = 0$, then it will also satisfy equation (ii).

$\therefore$ It follows that we can derive two independent solutions of equation (i) by solving the two subsidiary equations.

$$(E-\lambda_1)y_n = 0 \text{ and } (E-\lambda_2)y_n = 0$$

Their solutions are respectively,

$$y_n = c_1(\lambda_1)^n \text{ and } y_n = c_2(\lambda_2)^n$$

where, c_1, c_2 are arbitrary constants.

Thus the general solution of equation (i) is

$$y_n = c_1(\lambda_1)^n + c_2(\lambda_2)^n$$

Case II. If the roots are real and equal i.e. $\lambda_1 = \lambda_2$, then equation (ii) becomes

$$(E-\lambda_1)^2 y_n = 0$$

Let $\quad y_n = (\lambda_1)^n z_n \qquad \text{...(iv)}$

where z_n is a new dependent variable. Then equation (iv) takes the form

$$(\lambda_1)^{n+2} z_{n+2} - 2\lambda_1(\lambda_1)^{n+1} z_{n+1} + \lambda_1^2(\lambda_1)^n z_n = 0$$

or $\quad z_{n+2} - 2z_{n+1} + z_n = 0$

i.e. $\quad \Delta^2 z_n = 0$

$\therefore \quad z_n = c_1 + c_2 n$

where c_1, c_2 are arbitrary constants.

Thus solution of equation (i) becomes

$$y_n = (c_1 + c_2 n)(\lambda_1)^n$$

Case III. If the roots are imaginary, i.e. $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ then the solution of equation (i) is

$$y_n = c_1(\alpha + i\beta)^n + c_2(\alpha - i\beta)^n$$

Put $\alpha = r\cos\theta$ and $\beta = r\sin\theta$

$$= r^n\,[c_1(\cos n\theta + i\sin n\theta) + c_2(\cos n\theta - i\sin n\theta)]$$

$$= r^n\,[A_1\cos n\theta + A_2\sin n\theta]$$

where A_1, A_2 are arbitrary constants and

$$r = \sqrt{\alpha^2 + \beta^2},\ \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

(iii) In general, to solve the equation

$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \ldots + a_r y_n = 0$

where a's are constants.

(a) Write the equation in the symbolic form

$(E^r + a_1E^{r-1} + \ldots + a_r)y_n = 0$

(b) Write down the auxiliary equation

i.e. $E^r + a_1E^{r-1} + \ldots + a_r = 0$

and solve it for E.

(c) Write the solution as follows :

S.No.	*Roots of Auxiliary Equation*	*Solution i.e. Complementary Function*
(i)	Real and distinct roots $\lambda_1, \lambda_2, \lambda_3 \ldots$	$c_1(\lambda_1)^n + c_2(\lambda_2)^n + c_3(\lambda_3)^n + \ldots$
(ii)	Two real and equal roots	$(c_1 + c_2 n)(\lambda_1)^n + c_3(\lambda_3)^n + \ldots$
(iii)	Three real and equal roots $\lambda_1, \lambda_2, \lambda_3 \ldots$	$(c_1 + c_2 n + c_3 n^2)(y_1)^n + \ldots$
(iv)	A pair of imaginary roots $\alpha + i\beta, \alpha - i\beta, \ldots$	$r^n(c_1\cos n\theta + c_2\sin n\theta)$ where $r = \sqrt{\alpha^2+\beta^2}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

$$= 12 \frac{1}{4^2 - 7(4) + 10} 4^x$$

$$\text{P.I.} = 12 \frac{1}{-2} 4^x = -6(4)^x$$

Hence the complete solution is

$$y_x = c_1(2)^x + c_2(5)^x - 6(4)^x \quad \textbf{Ans.}$$

Example 4.16. *Solve* $y_{x+2} - 4y_{x+1} + 4y_x = 3.2^x + 5.4^x$.

Sol. Given equation in symbolic form is

$$(E^2 - 4E + 4)\, y_x = 3 \,.\, 2^x + 5 \,.\, 4^x$$

Its auxiliary equation is

$$E^2 - 4E + 4 = 0$$

$$\Rightarrow \quad E^2 - 2E - 2E + 4 = 0$$

$$\Rightarrow \quad (E-2)(E-2) = 0$$

$$\Rightarrow \quad E = 2, 2$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\, 2^x$$

and

$$\text{P.I.} = \frac{1}{E^2 - 4E + 4} (3 \,.\, 2^x + 5 \,.\, 4^x)$$

$$= 3 \frac{1}{E^2 - 4E + 4} \,.\, 2^x + 5 \,.\, \frac{1}{E^2 - 4E + 4} \,.\, 4^x$$

$$= 3 \,.\, \frac{1}{(E-2)^2} \,.\, 2^x + 5 \,.\, \frac{1}{16 - 16 + 4} \,.\, 4^x$$

$$= 3 \,.\, \frac{x(x-1)}{2!} \,.\, 2^{x-2} + \frac{5}{4} \,.\, 4^x$$

$$\text{P.I.} = 3x(x-1)\, 2^{x-3} + 5.4^{x-1}$$

Hence the complete solution is

$$y_x = (c_1 + c_2 x)\, 2^x + 3x(x-1)\, 2^{x-3} + 5.4^{x-1} \quad \textbf{Ans.}$$

Example 4.17. *Solve the equation* $y_{k+3} + 2y_{k+2} - 5y_{k+1} - 6y_k = 32$.

Sol. Given equation in symbolic form is

$$(E^3 + 2E^2 - 5E - 6)\, y_k = 32$$

Its auxiliary equation is

$$E^3 + 2E^2 - 5E - 6 = 0$$

$$\Rightarrow \quad (E+1)(E^2 + E - 6) = 0$$

$$\Rightarrow \quad (E+1)(E-2)(E+3) = 0$$

$$\Rightarrow \quad E = -1, 2, -3$$

$$\therefore \quad \text{C.F.} = c_1(-1)^k + c_2(2)^k + c_3(-3)^k$$

and

$$\text{P.I.} = \frac{1}{E^3 + 2E^2 - 5E - 6} \,.\, 32$$

$$= \frac{1}{1^3 + 2.1^2 - 5.1 - 6} \,.\, 32$$

$$\text{P.I.} = -\frac{32}{8} = -4$$

Hence the complete solution is

$$y_n = c_1(-1)^k + c_2(2)^k + c_3(-3)^k - 4 \quad \textbf{Ans.}$$

Example 4.18. *Solve* $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = \cos \alpha n$.

Sol. Given equation in symbolic form is

$$(E^2 - 2\cos\alpha E + 1)\, y_n = \cos\alpha n$$

Its auxiliary equation is

$$E^2 - 2\cos\alpha E + 1 = 0$$

$$E = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2}$$

$$= \cos\alpha \pm i\sin\alpha$$

$$\therefore \quad \text{C.F.} = (1)^n [c_1 \cos\alpha n + c_2 \sin\alpha n]$$

$$\text{P.I.} = \frac{1}{E^2 - 2E\cos\alpha + 1} \cos\alpha n$$

$$= \frac{1}{E^2 - E(e^{i\alpha} + e^{-i\alpha}) + 1} \left(\frac{e^{i\alpha n} + e^{-i\alpha n}}{2} \right)$$

$$= \frac{1}{E^2 - Ee^{i\alpha} - Ee^{-i\alpha} + e^{i\alpha} e^{-i\alpha}} \left(\frac{e^{i\alpha n} + e^{-i\alpha n}}{2} \right)$$

$$= \frac{1}{2} \left[\frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})} e^{i\alpha n} + \frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})} e^{-i\alpha n} \right]$$

Put $E = e^{i\alpha}$ in first term and $E = e^{-i\alpha}$ in second term, we get

$$= \frac{1}{2} \left[\frac{1}{(E - e^{i\alpha})(e^{i\alpha} - e^{-i\alpha})} e^{i\alpha n} + \frac{1}{(E - e^{-i\alpha})(e^{-i\alpha} - e^{i\alpha})} e^{-i\alpha n} \right]$$

$$= \frac{1}{4i\sin\alpha} \left[\frac{1}{(E - e^{i\alpha})} e^{i\alpha n} - \frac{1}{(E - e^{-i\alpha})} e^{-i\alpha n} \right]$$

$$= \frac{1}{4i\sin\alpha} [ne^{i\alpha(n-1)} - ne^{-i\alpha(n-1)}]$$

11. $A=\begin{bmatrix}1&2&3\\3&2&1\\0&0&2\end{bmatrix}, B=\begin{bmatrix}0&-2&-1\\-1&0&1\\1&-1&0\end{bmatrix}$

12. $X=\begin{bmatrix}-2&0\\-1&-3\end{bmatrix}, Y=\begin{bmatrix}2&1\\2&2\end{bmatrix}$

13. $X=\begin{bmatrix}2&0\\1&2\end{bmatrix}, Y=\begin{bmatrix}-1&-2\\2&2\end{bmatrix}$

14. $\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$

15. $2\begin{bmatrix}1&0\\0&1\end{bmatrix}+3\begin{bmatrix}0&1\\1&0\end{bmatrix}$

5.7 MULTIPLICATION OF MATRICES

If A and B are two matrices, then their product AB is possible only when the number of columns of matrix A is equal to the number of rows of the matrix B. Then the matrices A and B are said to be conformable for the multiplication AB.

In the multiplication AB, A is called the premultiplier and B the post multiplier.

Thus if A is a matrix of order $m \times n$ and B is of order $n \times p$, then their multiplication $AB = C$ will be of order $m \times p$. The $(i, k)^{th}$ element C_{ik} of the matrix C is obtained by taking i^{th} row of A and k^{th} column of B and then multiplying the corresponding elements and then adding i.e.,

if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$

then $C = [C_{ik}]_{m \times p}$

where
$$C_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \ldots + a_{in} b_{nk}$$
$$= \sum_{j=1}^{n} a_{ij} b_{jk}$$

Here j is called dummy suffix and
$$i = 1, 2, 3 \ldots m$$
$$k = 1, 2, 3, \ldots p$$

This multiplication operation is clear from the following illustration :

$$\begin{array}{cc} & A \\ i & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \end{array} \begin{array}{c} B \; k \\ \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1k} & \cdots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2k} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{j1} & b_{j2} & b_{j3} & \cdots & b_{jk} & \cdots & b_{jp} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nk} & \cdots & b_{np} \end{bmatrix}_{n \times p} \end{array}$$

$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1k} & \cdots & c_{1p} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2k} & \cdots & c_{2p} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i1} & c_{i2} & c_{i3} & \cdots & c_{ik} & \cdots & c_{ip} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & c_{m3} & \cdots & c_{mk} & \cdots & c_{mp} \end{bmatrix}_{m \times p}$$

5.7.1 Properties of Matrix Multiplication

5.7.1 Properties of Matrix Multiplication : Properties of matrix multiplication are as follows :

(i) Matrix Multiplication in General, is not Commutative : This is clear from the following arguments :

(a) We know that AB is only defined when the number of columns of A is equal to the number of rows in B. If AB is defined then it is not necessary that BA may also be defined.

For example, if A is 4×4 order matrix and B is 4×3 matrix, then AB is defined but BA is not defined as the number of columns in B is not equal to the number of rows in A. Hence $AB \neq BA$.

(b) If AB and BA are defined, it is not necessary that they are equal, because the orders of AB and BA may be different.

For example, if A is a 5×4 order matrix and B is a 4×5 order matrix, then the order of AB will be 5×5 where as that of BA will be 4×4. Hence $AB \neq BA$.

(c) If AB and BA both are defined and both are of the same order then it is not necessary that $AB = BA$, the fact follows from the following example :

Let $A = \begin{bmatrix}1&2\\0&3\end{bmatrix}$ and $B = \begin{bmatrix}0&1\\3&2\end{bmatrix}$

then
$$AB = \begin{bmatrix}1\times0+2\times3 & 1\times1+2\times2\\0\times0+3\times3 & 0\times1+3\times2\end{bmatrix}$$
$$= \begin{bmatrix}6&5\\9&6\end{bmatrix}$$

and
$$BA = \begin{bmatrix}0\times1+1\times0 & 0\times2+1\times3\\3\times1+2\times0 & 3\times2+2\times3\end{bmatrix}$$
$$= \begin{bmatrix}0&3\\3&12\end{bmatrix}$$

Here AB and BA both are defined and both are of same order 2×2. But the corresponding elements of AB and BA are not equal and hence $AB \neq BA$.

(d) But this does not mean that we can never have $AB = BA$. In some special cases this equality may hold.

For example, if A is a square matrix and B is the unit matrix of the same order as A, then $AI = IA = A$

$$= \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1.7+1.4+(-1)7 & 1.2+1.0+(-1).(-2) & 1.(-3)+1.(-4)+(-1)(-11) & 1(-1)+1.2+(-1).8 \\ 2.7+0.4+3.7 & 2.2+0.0+3.(-2) & 2.(-3)+0.(-4)+3.(-11) & 2(-1)+0.2+3.8 \\ 3.7+(-1).4+2.7 & 3.2+(-1).0+2(-2) & 3.(-3)+(-1).(-4)+2.(-11) & 3(-1)+(-1).2+2.8 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$$

$\therefore \quad (AB)C = A(BC)$ **Proved**

Example 5.25. *Find $A^2 - 4A - 5I$, if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.*

Sol. $A^2 = A.A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1.1+2.2+2.2 & 1.2+2.1+2.2 & 1.2+2.2+2.1 \\ 2.1+1.2+2.2 & 2.2+1.1+2.2 & 2.2+1.2+2.1 \\ 2.1+2.2+1.2 & 2.2+2.1+1.2 & 2.2+2.2+1.1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$ **Ans.**

Example 5.26. *If $A = [x \;\; y \;\; z]$, $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then find ABC.*

Sol. $AB = [x \;\; y \;\; z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

$$= [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz]$$

Hence, $ABC = [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$= [(ax + hy + gz)x + (hx + by + fz)y + (gx + fy + cz)z]$$

$$= [ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx]$$ **Ans.**

Example 5.27. *If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ where $i^2 = -1$, then prove that $(A+B)^2 = A^2 + B^2$.*

Sol. $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0+0 & 1-i \\ 1+i & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix}$$

$\therefore \quad (A+B)^2 = \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1-i^2 & 0 \\ 0 & 1-i^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad ...(i)$$

Now, $A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$B^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad A^2 + B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad ...(ii)$$

Hence, from equation (i) and (ii) we see that

$$(A+B)^2 = A^2 + B^2$$ **Proved**

Example 5.40. *If* $A=\begin{bmatrix}4 & 1\\5 & 8\end{bmatrix}$, *then prove that* $A+A^T$ *is symmetric matrix, when* A^T *is the transpose of the matrix A.*

Sol. Given,
$$A = \begin{bmatrix}4 & 1\\5 & 8\end{bmatrix}$$

then
$$A^T = \begin{bmatrix}4 & 5\\1 & 8\end{bmatrix}$$

$\therefore$
$$A+A^T = \begin{bmatrix}4 & 1\\5 & 8\end{bmatrix}+\begin{bmatrix}4 & 5\\1 & 8\end{bmatrix}$$
$$A+A^T = \begin{bmatrix}4+4 & 1+5\\5+1 & 8+8\end{bmatrix}$$
$$A+A^T = \begin{bmatrix}8 & 6\\6 & 16\end{bmatrix} \quad ...(i)$$
$$(A+A^T)^T = \begin{bmatrix}8 & 6\\6 & 16\end{bmatrix}$$
$$(A+A^T)^T = A+A^T \quad \text{[From equation (i)]}$$

Hence, $A+A^T$ is symmetric matrix. **Proved**

Example 5.41. *If* $A=\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$, *then prove that* $(A-A^T)$ *is skew symmetric, when* A^T *is the transpose of the matrix A.*

Sol. Given,
$$A = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$$

$\therefore$
$$A^T = \begin{bmatrix}2 & 4\\3 & 5\end{bmatrix}$$

$\therefore$
$$A-A^T = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}-\begin{bmatrix}2 & 4\\3 & 5\end{bmatrix}$$
$$A-A^T = \begin{bmatrix}2-2 & 3-4\\4-3 & 5-5\end{bmatrix}$$
$$A-A^T = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} \quad ...(i)$$
$$(A-A^T)^T = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$$
$$(A-A^T)^T = -\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

$\therefore$
$$(A-A^T)^T = -(A-A^T) \quad \text{[From equation (i)]}$$

Hence, $A-A^T$ is skew symmetric matrix. **Proved**

Example 5.42. *Express the matrix* $A=\begin{bmatrix}6 & 8 & 5\\4 & 2 & 3\\9 & 7 & 1\end{bmatrix}$ *as the sum of a symmetric and skew-symmetric matrix.*

Sol. Given,
$$A = \begin{bmatrix}6 & 8 & 5\\4 & 2 & 3\\9 & 7 & 1\end{bmatrix}$$

and
$$A' = \begin{bmatrix}6 & 4 & 9\\8 & 2 & 7\\5 & 3 & 1\end{bmatrix}$$

$\therefore$
$$A+A' = \begin{bmatrix}6 & 8 & 5\\4 & 2 & 3\\9 & 7 & 1\end{bmatrix}+\begin{bmatrix}6 & 4 & 9\\8 & 2 & 7\\5 & 3 & 1\end{bmatrix}$$
$$= \begin{bmatrix}12 & 12 & 14\\12 & 4 & 10\\14 & 10 & 2\end{bmatrix}$$
$$\frac{A+A'}{2} = \frac{1}{2}\begin{bmatrix}12 & 12 & 14\\12 & 4 & 10\\14 & 10 & 2\end{bmatrix}=\begin{bmatrix}6 & 6 & 7\\6 & 2 & 5\\7 & 5 & 1\end{bmatrix}$$
$$A-A' = \begin{bmatrix}6 & 8 & 5\\4 & 2 & 3\\9 & 7 & 1\end{bmatrix}-\begin{bmatrix}6 & 4 & 9\\8 & 2 & 7\\5 & 3 & 1\end{bmatrix}$$
$$= \begin{bmatrix}0 & 4 & -4\\-4 & 0 & -4\\4 & 4 & 0\end{bmatrix}$$
$$\frac{A-A'}{2} = \frac{1}{2}\begin{bmatrix}0 & 4 & -4\\-4 & 0 & -4\\4 & 4 & 0\end{bmatrix}=\begin{bmatrix}0 & 2 & -2\\-2 & 0 & -2\\2 & 2 & 0\end{bmatrix}$$

Sum of symmetric and skew-symmetric matrix is
$$A = \frac{A+A'}{2}+\frac{A-A'}{2}$$
$$= \begin{bmatrix}6 & 6 & 7\\6 & 2 & 5\\7 & 5 & 1\end{bmatrix}+\begin{bmatrix}0 & 2 & -2\\-2 & 0 & -2\\2 & 2 & 0\end{bmatrix} \quad \textbf{Ans.}$$

Answers

1. (i) $\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ 10 & 0 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & -6 & -9 \\ -9 & -3 & 6 \\ -5 & 3 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 6 & 6 & 3 \\ -3 & 15 & 3 \\ 3 & -6 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 3 & -5 \\ 0 & -6 & 4 \\ -3 & 3 & -1 \end{bmatrix}$ 3. $\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

6. $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin x & \cos x \end{bmatrix}$ 7. 0, 3

5.12 INVERSE MATRIX (RECIPROCAL OF A MATRIX)

Let A be a square matrix of order n. Then the matrix B of the same order n, if it exists, such that

$$AB = BA = I_n \qquad \text{...(i)}$$

is called the "inverse of A" or "reciprocal of A" and is denoted by A^{-1}, i.e.,

$$B = A^{-1}$$

Similarly, by virtue of equation (i), we can say that A is the inverse of B, i.e.,

$$A = B^{-1}$$

For example, let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

then
$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Similarly,
$$BA = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & -4+4 \\ \frac{3}{2}-\frac{3}{2} & 3-2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\therefore$
$$AB = BA = I_2$$

5.12.1 Theorem : The inverse of a matrix A, is $A^{-1} = \dfrac{Adj\,A}{|A|}$ if A is non-singular.

Proof : We know that
$$A(Adj\,A) = (Adj\,A)A = |A|\,I \qquad \text{...(i)}$$

where I is a unit matrix of same order as of A.

$\because A$ is non-singular, $|A| \neq 0$

Hence dividing equation (i) by $|A|$, we get
$$\frac{A(Adj\,A)}{|A|} = \left(\frac{Adj\,A}{|A|}\right)A = I \qquad \text{...(ii)}$$

$$\Rightarrow \quad A^{-1} = \frac{Adj\,A}{|A|}$$

5.12.2 Theorem : The inverse of a matrix is unique.

Proof : If possible, let B and C be two inverse of the same matrix A. Then by definition of inverse
$$AB = BA = I \qquad \text{...(i)}$$
and
$$AC = CA = I \qquad \text{...(ii)}$$

From equation (i) and (ii), we get
$$C = CI = C(AB) = (CA)B = IB = B$$
i.e.,
$$C = B$$

Hence, A^{-1} is unique.

5.12.3 Theorem : A square matrix A has an inverse if and only if $|A| \neq 0$, i.e., a square matrix A has an inverse if and only if it is non-singular.

Proof : Let B be the inverse of the matrix A, then
$$AB = I$$
$\therefore$
$$|AB| = |I|$$
or
$$|A||B| = I \qquad \text{...(i)}$$
$\therefore$
$$|A| \neq 0$$
Conversely let
$$|A| \neq 0$$
Again let
$$B = \frac{Adj\,A}{|A|}$$
then
$$AB = A\,\frac{Adj\,A}{|A|}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$
$$= 1(18-12) - 1(9-3) + 1(4-2)$$
$$= 6 - 6 + 2 = 2$$

$\because$ $|A| \neq 0$, hence A^{-1} is exist.

$$A_{11} = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 18 - 12 = 6$$

$$A_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9-3) = -6$$

$$A_{13} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4 - 2 = 2$$

$$A_{21} = -\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9-4) = -5$$

$$A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 9 - 1 = 8$$

$$A_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4-1) = -3$$

$$A_{31} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1$$

$$A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$$

$$A_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$$

$$Adj\,A = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\,A}{|A|}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -\frac{5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

From equation (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -\frac{5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ 14 \\ 36 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 - 35 + 18 \\ -18 + 56 - 36 \\ 6 - 21 + 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ $x = 1, y = 2, z = 3$ **Ans.**

Example 5.63. *Solve the following system of equations by matrix method :*

$$x + 2y + z = 8$$
$$x - 3y + 2z = 1$$
$$3x + y - z = 2$$

Sol. Given equations can be written in the matrix form as $AX = B$

where, $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 8 \\ 1 \\ 2 \end{bmatrix}$

$$X = A^{-1}B \qquad \text{...(i)}$$

$$|A| = 1(3-2) - 2(-1-6) + 1(1+9)$$
$$= 1 + 14 + 10 = 25$$

$\because$ $|A| \neq 0$, hence A^{-1} will exist.

Now, $A_{11} = \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} = 3 - 2 = 1$

$$A_{12} = -\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -(-1-6) = 7$$

$$A_{13} = \begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix} = 1 + 9 = 10$$

$$A_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -(-2-1) = 3$$

$$A_{22} = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1 - 3 = -4$$

$$A_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -(1-6) = 5$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= 1(18-12) - 1(9-3) + 1(4-2)$$

$$= 6 - 6 + 2 = 2$$

$\because$ $|A| \neq 0$, hence A^{-1} is exist.

$$A_{11} = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 18 - 12 = 6$$

$$A_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9-3) = -6$$

$$A_{13} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4 - 2 = 2$$

$$A_{21} = -\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9-4) = -5$$

$$A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 9 - 1 = 8$$

$$A_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4-1) = -3$$

$$A_{31} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1$$

$$A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$$

$$A_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$$

$$Adj\,A = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\,A}{|A|}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -\frac{5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

From equation (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -\frac{5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 6 \\ 14 \\ 36 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 - 35 + 18 \\ -18 + 56 - 36 \\ 6 - 21 + 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ $x = 1, y = 2, z = 3$ **Ans.**

Example 5.63. *Solve the following system of equations by matrix method :*

$$x + 2y + z = 8$$
$$x - 3y + 2z = 1$$
$$3x + y - z = 2$$

Sol. Given equations can be written in the matrix form as $AX = B$

where, $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 8 \\ 1 \\ 2 \end{bmatrix}$

$$X = A^{-1}B \qquad \text{...(i)}$$

$$|A| = 1(3-2) - 2(-1-6) + 1(1+9)$$

$$= 1 + 14 + 10 = 25$$

$\because$ $|A| \neq 0$, hence A^{-1} will exist.

Now, $A_{11} = \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} = 3 - 2 = 1$

$$A_{12} = -\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -(-1-6) = 7$$

$$A_{13} = \begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix} = 1 + 9 = 10$$

$$A_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -(-2-1) = 3$$

$$A_{22} = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1 - 3 = -4$$

$$A_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -(1-6) = 5$$

then
$$[A:B] = \begin{bmatrix} 1 & -2 & -1 & : & 5 \\ 1 & 8 & -3 & : & -1 \\ 2 & 1 & -3 & : & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & : & 5 \\ 0 & 10 & -2 & : & -6 \\ 0 & 5 & -1 & : & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$[A:B] \sim \begin{bmatrix} 1 & -2 & -1 & : & 5 \\ 0 & 10 & -2 & : & -6 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad R_3 \to R_3 - \frac{1}{2}R_2$$

$\Rightarrow$ Rank of $[A:B] = 2$ and rank of $A = 2$

Hence, the given system is consistent. **Ans.**

But number of unknowns = 3

Rank of $[A:B]$ = Rank of $A < n$

Hence the given system has infinite number of solution.

In this case, we shall assign arbitrary values to $n - \rho(A)$ i.e., $3 - 2 = 1$ unknown and remaining two unknown shall be found in terms of these.

The given system can be put as

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 10 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x - 2y - z &= 5 \\ 10y - 2z &= -6 \\ 0 &= 0 \end{aligned}$$

Now let us choose $z = r$ so that $10y = 2r - 6$

$$\Rightarrow \quad y = \frac{2r-6}{10} = \frac{r-3}{5}$$

and
$$x = 2y + z + 5$$
$$= 2\left(\frac{r-3}{5}\right) + r + 5 = \frac{7r+19}{5}$$

$$x = \frac{7r+19}{5}, y = \frac{r-3}{5}, z = r$$ **Ans.**

Example 5.75. *Show that the following system of equations are consistent and solve them :*

$$\begin{aligned} x + y + z &= 6 \\ x - y + 2z &= 5 \\ 3x + y + z &= 8 \\ 2x - 2y + 3z &= 7 \end{aligned}$$

Sol. The given in the matrix form $AX = B$ can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \end{bmatrix}$$

The augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \\ 2 & -2 & 3 & : & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \\ 0 & -4 & 1 & : & -5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -3 & : & -9 \\ 0 & 0 & 5 & : & 15 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -3 & : & -9 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad R_4 \to R_4 + \frac{5}{3}R_3$$

Now, rank of $[A:B] = 3$ and rank of $A = 3$

$\Rightarrow$ Rank of $[A:B]$ = Rank of A = Number of unknowns

Therefore the given equations are consistent. **Proved**

$\Rightarrow$ The system has a unique solution. At this stage system is matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -9 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x + y + z &= 6 \\ -2y + z &= -1 \\ -3z &= -9 \end{aligned}$$

$\therefore$ $z = 3$

Now $-2y + 3 = -1$

or $-2y = -4$

$\therefore$ $y = 2$

and $x + y + z = 6$

$x + 2 + 3 = 6$

$x = 1$

Ans.

$\therefore$ $\mathbf{x = 1, y = 2, z = 3}$

$$\therefore \int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\,dx = \int \frac{1}{t}\,\frac{dt}{2(b-a)}$$

$$= \frac{1}{2(b-a)}\int \frac{1}{t}\,dt$$

$$= \frac{1}{2(b-a)}\log_e t + c$$

$$= \frac{1}{2(b-a)}\log_e (a\cos^2 x + b\sin^2 x) + c \qquad \textbf{Ans.}$$

(ii) $\int \frac{x^2 \tan^{-1} x^3}{1+x^6}\,dx$

Let $\tan^{-1} x^3 = t$

$$\frac{3x^2}{1+(x^3)^2}\,dx = dt$$

$$\frac{x^2}{1+x^6}\,dx = \frac{dt}{3}$$

$$\int \frac{x^2 \tan^{-1} x^3}{1+x^6}\,dx = \int t\,\frac{dt}{3}$$

$$= \frac{1}{3}\int t\,dt$$

$$= \frac{1}{3}\cdot\frac{t^2}{2} + c = \frac{t^2}{6} + c$$

$$= \frac{(\tan^{-1} x^3)^2}{6} + c \qquad \textbf{Ans.}$$

(iii) $\int \sin^5 x\,dx$

$$= \int \sin^4 x \sin x\,dx$$

$$= \int (\sin^2 x)^2 \sin x\,dx$$

$$= \int (1-\cos^2 x)^2 \sin x\,dx$$

Let $\cos x = t$

$$-\sin x\,dx = dt$$

$$\int \sin^5 x\,dx = \int (1-t^2)^2(-dt)$$

$$= -\int (1+t^4-2t^2)\,dt$$

$$= -t - \frac{t^5}{5} + 2\frac{t^3}{3} + c$$

$$= -\cos x - \frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} + c \qquad \textbf{Ans.}$$

(iv) $\int \frac{(x+1)(x+\log_e x)^3}{2x}\,dx$

Let $x + \log_e x = t$

$$\left(1+\frac{1}{x}\right)dx = dt$$

$$\frac{x+1}{x}\,dx = dt$$

$$\therefore \int \frac{(x+1)(x+\log_e x)^3}{2x}\,dx = \frac{1}{2}\int t^3\,dt$$

$$= \frac{1}{2}\frac{t^4}{4} + c$$

$$= \frac{t^4}{8} + c$$

$$= \frac{(x+\log_e x)^4}{8} + c \qquad \textbf{Ans.}$$

EXERCISE 6.2

Integrate the following functions with respect to x :

1. (i) $e^x \cos e^x$ (ii) $x^2 \sin x^3$
   (iii) $\frac{\sin(\log_e x)}{x}$ (iv) $\frac{\sin(\tan^{-1} x)}{1+x^2}$
2. (i) $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$ (ii) $\frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$
   (iii) $\frac{1}{x+\sqrt{x}}$ (iv) $\frac{\sin x \cos x}{1+\sin^4 x}$
3. (i) $\frac{x}{1+x^4}$ (ii) $\frac{x^2}{1+x^6}$
   (iii) $\frac{e^{\log_e x}}{x}$ (iv) $\frac{1}{x\sqrt{1-(\log_e x)^2}}$
4. (i) $\frac{\sin\sqrt{x}}{\sqrt{x}}$ (ii) $\frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}}$
   (iii) $(e^x+1)^3 e^x$ (iv) $(e^x+a)^n e^x$
5. (i) $\sin^5 x \cos x$ (ii) $\sec^5 x \tan x$
   (iii) $\cot x \,\mathrm{cosec}^3 x$ (iv) $(2+3\sin x)^3 \cos x$

11. (i) $\frac{1}{4}(\tan^{-1} x^2)^2 + c$ (ii) $\sin(\tan^{-1} x) + c$

12. (i) $-\frac{1}{4}\cos^4 x^2 + c$ (ii) $\frac{1}{b(n+1)}(a + be^x)^{n+1} + c$

13. (i) $-\frac{1}{3}\cot^3 x + c$ (ii) $\log[\log(\sec x + \tan x)] + c$

14. (i) $\log\frac{e^x}{e^x + 1} + c$ (ii) $-\frac{2}{3}\cos 3x + c$

15. (i) $\frac{1}{5}\sin^{-1} x^5 + c$ (ii) $\frac{1}{4}\sin^{-1} x^4 + c$

6.4.1 Trigonometrical Transformation : Sometimes an integration can be done easily by dividing an integrand into the sum or difference of two functions. This method is generally applied to find out the integral of trigonometric product.

ILLUSTRATIVE EXAMPLES

Example 6.17. Evaluate :

(i). $\int \sin^3 x\, dx$ *(ii)* $\int \sin 2x \cos 3x\, dx$

Sol. (i) $\int \sin^3 x\, dx$

We know that by trigonometry

$$\sin 3x = 3\sin x - 4\sin^3 x$$

or

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

Now

$$\int \sin^3 x\, dx = \int \frac{1}{4}(3\sin x - \sin 3x)\, dx$$

$$= \frac{3}{4}\int \sin x\, dx - \frac{1}{4}\int \sin 3x\, dx$$

$$= \frac{3}{4}(-\cos x) - \frac{1}{4}\left(-\frac{\cos 3x}{3}\right) + c$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + c \quad \textbf{Ans.}$$

(ii) $\int \sin 2x \cos 3x\, dx$

We know that

$$2\sin A\cos B = \sin(A + B) + \sin(A - B)$$

Now

$$\int \sin 2x \cos 3x\, dx = \frac{1}{2}\int 2\sin 2x\cos 3x\, dx$$

$$= \frac{1}{2}\int [\sin 5x + \sin(-x)]\, dx$$

$$= \frac{1}{2}\int \sin 5x\, dx - \frac{1}{2}\int \sin x\, dx$$

$$= \frac{1}{2}\left[-\frac{\cos 5x}{5} - (-\cos x)\right] + c$$

$$= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + c \quad \textbf{Ans.}$$

Example 6.18. *Evaluate the following :*

(i) $\int \cos 2x \cos 4x \cos 6x\, dx$ *(ii)* $\int \sin^2(2x + 5)\, dx$

(iii) $\int \sin^3 x \cos^3 x\, dx$ *(iv)* $\int \cos^4 2x\, dx$

Sol. (i) $\int \cos 2x \cos 4x \cos 6x\, dx$

$$= \frac{1}{4}\int (2\cos 2x\cos 4x)(2\cos 6x)\, dx$$

$$= \frac{1}{4}\int (\cos 6x + \cos 2x)(2\cos 6x)\, dx$$

$$= \frac{1}{4}\int (2\cos^2 6x + 2\cos 6x\cos 2x)\, dx$$

$$= \frac{1}{4}\int [(1 + \cos 12x) + (\cos 8x + \cos 4x)]\, dx$$

$$= \frac{1}{4}\int [\cos 12x + \cos 8x + \cos 4x + 1]\, dx$$

$$= \frac{1}{4}\left[\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4} + x\right] + c \quad \textbf{Ans.}$$

(ii) $\int \sin^2(2x + 5)\, dx$

$$= \int \frac{1}{2}\{1 - \cos[2(2x + 5)]\}\, dx$$

$$\left[\because \sin^2 x = \frac{1}{2}(1 - \cos 2x)\right]$$

$$= \frac{1}{2}\int [1 - \cos(4x + 10)]\, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(4x + 10)}{4}\right] + c$$

$$= \frac{1}{2}x - \frac{\sin(4x + 10)}{8} + c \quad \textbf{Ans.}$$

(iii) $\int \sin^3 x \cos^3 x\, dx$

$$= \frac{1}{8}\int (2\sin x\cos x)^3\, dx$$

$$= \frac{1}{8}\int \sin^3 2x\, dx$$

(ii) If both the functions have standard result of integration then take that function as first whose differentiation vanishes.

(iii) If the differentiation of any functions is not zero, then we may take any function as first function.

6.6.1 Trigonometrical Substitution in Integration by Parts : Sometimes the given integral is transformed by the proper trigonometrical substitution before doing the integration by parts. In such cases the application of formulae of integration by parts will ease the process of integration.

ILLUSTRATIVE EXAMPLES

Example 6.24. Integrate the following :

(i) $\int xe^x\,dx$ (ii) $\int x^2 \sin x\,dx$

(iii) $\int x\cos^2 x\,dx$ (iv) $\int x^n \log x\,dx$

Sol. (i) $\int xe^x\,dx$

Let $I = \int xe^x\,dx$

$$= x\int e^x\,dx - \int\left[\frac{d}{dx}x\int e^x\,dx\right]dx$$

$$= xe^x - \int[1.e^x]\,dx$$

$$= xe^x - e^x + c$$

$$= e^x(x-1) + c$$ **Ans.**

(ii) $\int x^2 \sin x\,dx$

Let $I = \int x^2 \sin x\,dx$

$$= x^2\int \sin x\,dx - \int\left[\frac{d}{dx}x^2\int \sin x\,dx\right]dx$$

$$= x^2(-\cos x) - \int[2x(-\cos x)]\,dx$$

$$= -x^2\cos x + 2\int x\cos x\,dx$$

$$= -x^2\cos x + 2\left[x\int\cos x\,dx - \int\left(\frac{d}{dx}x\int\cos x\,dx\right)dx\right]$$

$$= -x^2\cos x + 2[x\sin x - \int(1.\sin x)\,dx]$$

$$= -x^2\cos x + 2[x\sin x - (-\cos x)]$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

$$= (2-x^2)\cos x + 2x\sin x + c$$ **Ans.**

(iii) $\int x\cos^2 x\,dx$

Let $I = \int x\cos^2 x\,dx$

$$= \int x\left(\frac{1+\cos 2x}{2}\right)dx$$

$$= \frac{1}{2}\int x(1+\cos 2x)\,dx$$

$$= \frac{1}{2}[\int x\,dx + \int x\cos 2x\,dx]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \left\{x.\left(\frac{\sin 2x}{2}\right) - \int 1.\frac{\sin 2x}{2}\,dx\right\}\right]$$

$$= \frac{1}{4}\left[x^2 + x\sin 2x - \int \sin 2x\,dx\right]$$

$$= \frac{1}{4}\left[x^2 + x\sin 2x - \left(-\frac{\cos 2x}{2}\right)\right] + c$$

$$= \frac{1}{4}\left[x^2 + x\sin 2x + \frac{1}{2}\cos 2x\right] + c$$ **Ans.**

(iv) $\int x^n \log x\,dx$

Let $I = \int x^n \log x\,dx$

$$= \log x.\frac{x^{n+1}}{n+1} - \int\frac{1}{x}\frac{x^{n+1}}{n+1}\,dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^n\,dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2} + c$$ **Ans.**

Example 6.25. Evaluate the following :

(i) $\int x\log_e x\,dx$ (ii) $\int\frac{\log_e x}{x^3}\,dx$

(iii) $\int x^n(\log_e x)^2\,dx$ (iv) $\int\frac{xe^x}{(1+x)^2}\,dx$

Sol. (i) $\int x\log_e x\,dx$

Let $I = \int x\log_e x\,dx$

$$= \log_e x.\frac{x^2}{2} - \int\left[\frac{1}{x}.\frac{x^2}{2}\right]dx$$

$$= \log(1-x) + \frac{1}{2}\log(1+x^2) + \tan^{-1} x + c \qquad \text{Ans.}$$

Example 6.31. *Evaluate :*

(i) $\int \frac{dx}{1+x+x^2+x^3}$ *(ii)* $\int \frac{dx}{x^3-1}$

Sol. (i) $\int \frac{dx}{1+x+x^2+x^3} = \int \frac{dx}{(1+x)+x^2(1+x)} = \int \frac{dx}{(1+x)(1+x^2)}$

Let $\frac{1}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$ ...(i)

$$1 = A(1+x^2) + (Bx+C)(1+x) \qquad \text{...(ii)}$$

Put $x = -1$ in equation (ii),

$$A = \frac{1}{2}$$

Now, equating the coefficients of like power of x in equation (ii), we get

$$A + B = 0 \Rightarrow \frac{1}{2} + B = 0 \Rightarrow B = -\frac{1}{2}$$

and $A + C = 1 \Rightarrow C = 1 - A = 1 - \frac{1}{2} = \frac{1}{2}$

$$\therefore \frac{1}{(1+x)(1+x^2)} = \frac{1}{2(1+x)} + \frac{(1-x)}{2(1+x^2)}$$

$$= \frac{1}{2(1+x)} - \frac{x}{2(1+x^2)} + \frac{1}{2(1+x^2)}$$

$$\int \frac{dx}{(1+x)(1+x^2)} = \frac{1}{2}\int \frac{dx}{(1+x)} + \frac{1}{2}\int \frac{dx}{1+x^2} - \frac{1}{4}\int \frac{2x}{1+x^2}\,dx$$

$$= \frac{1}{2}\log(1+x) + \frac{1}{2}\tan^{-1} x + c - \frac{1}{4}\log(1+x^2)$$

Ans.

(ii) $\int \frac{dx}{x^3-1}$

Here $x^3 - 1 = (x-1)(x^2+x+1)$

Let $\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$ ...(i)

$$1 = A(x^2+x+1) + (Bx+C)(x-1) \qquad \text{...(ii)}$$

Put $x = 1$ in equation (i), we get

$$A = \frac{1}{3}$$

Put $x = 0$ in equation (i), we get

$$1 = A - C \Rightarrow C = \frac{1}{3} - 1 = -\frac{2}{3}$$

Equating the coefficients of like powers of x, in equation (ii), we get

$$A + B = 0 \Rightarrow B = -\frac{1}{3}$$

Putting the value of A, B and C in equation (i), we get

$$\therefore \frac{1}{x^3-1} = \frac{1}{3(x-1)} - \frac{1}{3}\frac{(x+2)}{x^2+x+1}$$

$$\int \frac{dx}{x^3-1} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{(x+2)}{x^2+x+1}\,dx$$

$$= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{\frac{1}{2}(2x+1) + \frac{3}{2}}{x^2+x+1}\,dx$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{3}\cdot\frac{1}{2}\int \frac{(2x+1)}{x^2+x+1}\,dx - \frac{1}{3}\cdot\frac{3}{2}\int \frac{dx}{x^2+x+1}$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(2x+1) - \frac{1}{2}\int \frac{dx}{x^2+\frac{1}{4}+x+1-\frac{1}{4}}$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(2x+1) - \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(2x+1) - \frac{1}{2}\cdot\frac{1}{\sqrt{3}/2}\cdot\tan^{-1}\left[\frac{\left(x+\frac{1}{2}\right)}{\sqrt{3}/2}\right]$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(2x+1) - \frac{1}{\sqrt{3}}\cdot\tan^{-1}\frac{(2x+1)}{\sqrt{3}}$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + c \qquad \text{Ans.}$$

Example 6.32. *Evaluate :*

(i) $\int \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$ *(ii)* $\int \frac{dx}{1+3e^x+2e^{2x}}$

(iii) $\int \frac{\sec^2 x\,dx}{(\tan x+1)(\tan x+2)}$

Sol. (i) $\int \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$

Let $I = \int \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$

and $\sin x = t \Rightarrow \cos x\, dx = dt$

$\therefore$ $$I = \int \frac{dt}{(1+t)(2+t)}$$

Breaking into partial fraction

$$= \int \left[\frac{1}{1+t} - \frac{1}{2+t}\right] dt$$

$$= \int \frac{dt}{1+t} - \int \frac{dt}{2+t}$$

$$= \log(1+t) - \log(2+t)$$

$$= \log\left(\frac{1+t}{2+t}\right) + c$$

$$= \log\left(\frac{1+\sin x}{2+\sin x}\right) + c \qquad \textbf{Ans.}$$

(ii) $\displaystyle\int \frac{dx}{1+3e^x+2e^{2x}}$

Let $$I = \int \frac{dx}{1+3e^x+2e^{2x}}$$

and $e^x = t \Rightarrow e^x\, dx = dt \Rightarrow dx = \frac{1}{t}\, dt$

$\therefore$ $$I = \int \frac{1}{1+3t+2t^2} \cdot \frac{dt}{t} = \int \frac{dt}{t(t+1)(1+2t)}$$

$$= \int \left[\frac{1}{t} + \frac{1}{t+1} - \frac{4}{1+2t}\right] dt$$

$$= \log t + \log(t+1) - 4 \cdot \frac{1}{2} \log(1+2t) + c$$

$$= \log e^x + \log(e^x+1) - 2\log(1+2e^x) + c \qquad \textbf{Ans.}$$

(iii) $\displaystyle\int \frac{\sec^2 x\, dx}{(\tan x+1)(\tan x+2)}$

Let $$I = \int \frac{\sec^2 x\, dx}{(\tan x+1)(\tan x+2)}$$

$\therefore$ $\tan x + 1 = t \Rightarrow \sec^2 x\, dx = dt$
$\tan x + 2 = t + 1$

$$I = \int \frac{1}{t(t+1)}\, dt = \int \left[\frac{1}{t} - \frac{1}{t+1}\right] dt$$

$$= \int \frac{1}{t}\, dt - \int \frac{1}{t+1}\, dt = \log t - \log(t+1) + c$$

$$= \log(\tan x + 1) - \log(\tan x + 2) + c$$

$$= \log \frac{(\tan x + 1)}{(\tan x + 2)} + c \qquad \textbf{Ans.}$$

EXERCISE 6.6

Evaluate the following functions :

1. (i) $\displaystyle\int \frac{3x\, dx}{(x-2)(x+1)}$ (ii) $\displaystyle\int \frac{dx}{x^2-5x+6}$
2. (i) $\displaystyle\int \frac{x^2+1}{x(x^2-1)}\, dx$ (ii) $\displaystyle\int \frac{x^2}{x^2+7x+2}\, dx$
3. (i) $\displaystyle\int \frac{1}{(x+1)^2(x-1)}\, dx$ (ii) $\displaystyle\int \frac{x\, dx}{(x-1)^2(x+2)}$
4. (i) $\displaystyle\int \frac{3x+1}{(x-1)^3(x+1)}\, dx$ (ii) $\displaystyle\int \frac{dx}{x^3-x^2-x+1}$
5. (i) $\displaystyle\int \frac{dx}{(x-1)(x^2+4)}$ (ii) $\displaystyle\int \frac{(x-1)\, dx}{(x+1)(x^2+1)}$
6. (i) $\displaystyle\int \frac{dx}{(x^4-1)}$ (ii) $\displaystyle\int \frac{x^2\, dx}{(x-1)^2(x^2+1)}$
7. (i) $\displaystyle\int \frac{dx}{\sin x(3+2\cos x)}$ (ii) $\displaystyle\int \frac{dx}{(e^x-1)^2}$
8. (i) $\displaystyle\int \frac{\sin\theta\cos\theta\, d\theta}{\cos^2\theta - \cos\theta - 2}$ (ii) $\displaystyle\int \frac{x\, dx}{(x^2-a^2)(x^2-b^2)}$
9. (i) $\displaystyle\int \frac{dx}{x(1+\log x)(3+\log x)}$ (ii) $\displaystyle\int \frac{e^x\, dx}{e^{2x}+4e^x+3}$

Answers

1. (i) $2\log(x-2) + \log(x+1) + c$
   (ii) $\log(x-3) - \log(x-2) + c$
2. (i) $\log\left(\frac{x^2-x}{x}\right) + c$
   (ii) $\frac{1}{2}x^2 - 7x + 27\log(x+3) + 64\log(x+4) + c$
3. (i) $\frac{1}{4}\log\frac{x-1}{x+1} + \frac{1}{2(x+1)} + c$
   (ii) $-\frac{1}{4}\log(x+2) + \frac{1}{4}\log(x-1) - \frac{1}{4(x-1)} + c$
4. (i) $\frac{5}{8}\log\frac{x+1}{x-1} - \frac{5}{4(x-1)} - \frac{1}{2(x-1)^2} + c$

(ii) $\frac{1}{4}\log(x+1)-\frac{1}{4}\log(x-1)-\frac{1}{2(x-1)}+c$

5. (i) $\frac{1}{3}\log(x-1)-\frac{1}{10}\log(x^2+4)-\frac{1}{10}\tan^{-1}\frac{x}{2}+c$

(ii) $-\log(x+1)+\frac{1}{2}\log(x^2+1)+c$

6. (i) $\frac{1}{4}\log\frac{x-1}{x+1}-\frac{1}{2}\tan^{-1}x+c$

(ii) $\frac{1}{2}\log(x-1)-\frac{1}{2(x-1)}-\frac{1}{4}\log(x^2+1)+c$

7. (i) $-\frac{1}{10}\log(1-\cos x)+\frac{1}{2}\log(1+\cos x)-\frac{2}{5}\log(3+2\cos x)$

(ii) $\log e^x-\log(e^x-1)-\frac{1}{e^x-1}+c$

8. (i) $-\frac{2}{3}\log(\cos\theta-2)-\frac{1}{3}\log(\cos\theta+1)+c$

(ii) $\frac{1}{2(a^2-b^2)}\log\frac{x^2-a^2}{x^2-b^2}+c$

9. (i) $\frac{1}{2}[\log(1+\log x)-\log(3+\log x)]+c$

(ii) $\frac{1}{2}[\log(e^x+1)-\log(e^x+3)]+c$

6.8 INTEGRATION OF THE FORM AND REDUCTION FORMULA

A formula which connects a given integral to another integral of same type but of lower degree and easier to integrate is called the reduction formula. The reduction formulas are generally developed by the method of integration by parts. The difficult, complicated unintegrable functions by the repeated application of the same reduction formula, are reduced to a standard form which can be easily and directly integrated.

(i) Integration of $\sin^n x$ When n is Odd Positive Integer : Let $n = 2p+1$, where p is some positive integer then the above integral can be evaluated by the substitution of $\cos x = t$.

$$\begin{aligned}\int \cos^n x\,dx &= \int \sin^{2p+1} x\,dx = \int \sin^{2p} x \sin x\,dx\\ &= \int (1-\cos^2 x)^p \sin x\,dx\\ &= \int (1-t^2)^p(-dt) \qquad \text{[where } \cos x = t]\\ &= -\int (1-t^2)^p\,dt\end{aligned}$$

This integral can be easily evaluated by expanding $(1-t^2)^p$ using binomial theorem.

(ii) Integration of $\cos^n x$ When n is Odd Positive Integer : Let $n = 2p+1$, where p is some positive integer then the above integral can be evaluated by the substitution of $\sin x = t$.

$$\begin{aligned}\int \sin^n x\,dx &= \int \cos^{2p+1} x\,dx\\ &= \int \cos^{2p} x \,.\, \cos x\,dx \qquad \text{[where } \sin x = t]\\ &= \int (1-\sin^2 x)^p \cos x\,dx = \int (1-t^2)^p\,dt\end{aligned}$$

Now the integral can be evaluated by expanding $(1-t^2)^p$ with the help of binomial theorem.

(iii) Integration of $\sin^m x\cos^n x$: Following three cases are possible :

Case I. When m or n is an odd positive integer. If m is odd, then let $m = 2p+1$, where p is some positive integer. Then for all values of n, we have

$$\begin{aligned}\int \sin^m x\cos^n x\,dx &= \int \sin^{2p+1} x\cos^n x\,dx\\ &= \int \sin^{2p} x\cos^n x \,.\, \sin x\,dx\\ &= \int (1-\cos^2 x)^p\cos^n x\sin x\,dx\\ &= \int (1-t^2)^p t^n(-dt) \qquad \text{[where } \cos x = t]\end{aligned}$$

Now the integral can be evaluated by expanding $(1-t^2)^p$ with the help of binomial theorem. Hence when m is an odd positive integer then the integration of $\sin^m x\cos^n x$ is evaluated by substituting $\cos x = 1$.

Similarly when n is odd positive integer then the above integral is evaluated by substituting $\sin x = t$. But when m and n both are odd positive integers then the above integral can be evaluated by the substitution of $\cos x = t$ or $\sin x = t$.

Case II. When $m+n$ is a negative even integer.

Let $m+n = -2p$, where p is some positive integer. In this case the given integrand is converted in terms of $\tan x$ and $\sec x$ and then the given integral is evaluated by substituting $\tan x = t$.

$$\begin{aligned}\int \sin^m x\cos^n x\,dx &= \int \frac{\sin^m x}{\cos^m x}\cos^{m+n} x\,dx\\ &= \int \tan^m x\cos^{-2p} x\,dx\\ &= \int \tan^m x \,.\, \sec^{2p} x\,dx\\ &= \int \tan^m x\sec^{2p-2} x\sec^2 x\,dx\\ &= \int t^m(1+t^2)^{p-1}\,dt \qquad \text{[where } \tan x = t]\end{aligned}$$

Now the integral can be evaluated by expanding $(1+t^2)^{p-1}$ with the help of binomial theorem.

6.8.1 Reduction Formula for $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$

Let $I_n = \int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx$

Now integrating by parts, taking $\sin x$ as second function, we have

$$= \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n$$

or $$nI_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$$

or $$I_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} I_{(n-2)}$$

$$\therefore \int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \quad ...(i)$$

Similarly

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \quad ...(ii)$$

Equation (i) and (ii) are required reduction formula.

6.8.2 Reduction Formula for $\int \tan^n x\,dx$

$$\int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx$$

$$= \int \tan^{n-2} x(\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

or $$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$$

Similarly

$$\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

$$\int \operatorname{cosec}^n x\,dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x\,dx$$

6.8.3 Reduction Formula for $\int \sin^m x \cos^n x\,dx$

Let $$I_{(m,n)} = \int \sin^m x \cos^n x\,dx = \int \sin^{m-1} x \cos^n x \sin x\,dx$$

$$= \sin^{m-1} x\left[-\frac{\cos^{n+1} x}{n+1}\right] - \int (m-1)\sin^{m-2} x \cos x\left(\frac{-\cos^{n+1} x}{n+1}\right)dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n+2} x\,dx \quad ...(i)$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x(1-\sin^2 x)\,dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{(m-2,n)} - \frac{m-1}{n+1} I_{(m,n)}$$

or $$I_{(m,n)}\left(1 + \frac{m-1}{n+1}\right) = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{(m-2,n)}$$

or $$I_{(m,n)} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{(m-2,n)} \quad ...(ii)$$

The reduction formula for $\int \sin^m x \cos^n\,dx$ can be found in the following five forms :

(i) We may write

$$I_{(m,n)} = \int \cos^{n-1} x(\sin^m x \cos x)\,dx \quad ...(iii)$$

Now integrating by parts and proceeding as above, we get

$$I_{(m,n)} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{(m,n-2)} \quad ...(iv)$$

(ii) Substituting $m+2$ for m in equation (ii) and transposing, we get

$$I_{(m,n)} = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} I_{(m+2,n)} \quad ...(v)$$

(iii) Substituting $n+2$ for n in equation (iv) and transposing, we get

$$I_{(m,n)} = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} I_{(m,n+2)} \quad ...(vi)$$

(iv) From equation (i), we have

$$I_{(m,n)} = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{(m-2,n+2)} \quad ...(vii)$$

(v) The following reduction formula is obtained just after the integrating by parts in the deduction of equation (iv)

$$I_{(m,n)} = \frac{\sin^{m+1} x \cos^{n-1} x}{n+1} + \frac{n-1}{m+1} I_{(m+2,n-2)}$$

ILLUSTRATIVE EXAMPLES

Example 6.33. *Evaluate the following :*

(i) $\int \cos^3 x\,dx$ (ii) $\int \sin^2 3x\,dx$

(iii) $\int \sin^4 x \cos^3 x\,dx$ (iv) $\int \sin^5 x \cos^4 x\,dx$

Sol. (i) $\int \cos^3 x\,dx$

$$\int \cos^3 x\,dx = \int \cos^2 x \,.\, \cos x\,dx$$

(iii) $\int_1^2 \frac{3x^2}{1+x^3}dx$ (iv) $\int_0^1 \frac{x\tan^{-1}x^2}{1+x^4}dx$

Sol. (i) $\int_0^{\pi/2} \cos x\, dx$

Let, $I = \int_0^{\pi/2} \cos x\, dx$

$= \left[\sin x\right]_0^{\pi/2}$

$= \sin\frac{\pi}{2} - \sin 0$

$= 1 - 0$

$\int_0^{\pi/2} \cos x\, dx = 1$ Ans.

(ii) $\int_0^{\pi/2} \sqrt{1+\sin x}\, dx$

Let $I = \int_0^{\pi/2} \sqrt{1+\sin x}\, dx$

$= \int_0^{\pi/2} \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}$

$= \int_0^{\pi/2} \sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}$

$= \int_0^{\pi/2} \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)dx$

$= \left[2\sin\frac{x}{2} - 2\cos\frac{x}{2}\right]_0^{\pi/2}$

$= 2\left[\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) - (\sin 0 - \cos 0)\right]$

$= 2\left[\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - (0-1)\right]$

$= 2[0+1]$

$= 2$ Ans.

(iii) $\int_1^2 \frac{3x^2}{1+x^3}dx$

Let $I = \int_1^2 \frac{3x^2\, dx}{1+x^3}$

and $1+x^3 = t$

$3x^2\, dx = dt$

If $x = 1 \Rightarrow t = 2$

$x = 2 \Rightarrow t = 9$

Now $I = \int_2^9 \frac{dt}{t}$

$= \left[\log t\right]_2^9$

$= \log 9 - \log 2$

$= \log\frac{9}{2}$ Ans.

(iv) $\int_0^1 \frac{x\tan^{-1}x^2}{1+x^4}dx$

Let $I = \int_0^1 \frac{x\tan^{-1}x^2}{1+x^4}dx$

and $x^2 = t$

$2x\, dx = dt$

If $x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$I = \int_0^1 \frac{1}{2}\frac{\tan^{-1}t}{1+t^2}dt$

Again let $\tan^{-1}t = u$

$\frac{1}{1+t^2}dt = du$

If $t = 0 \Rightarrow u = 0$

$t = 1 \Rightarrow u = \tan^{-1}(1) = \frac{\pi}{4}$

Now $I = \frac{1}{2}\int_0^{\pi/4} u\, du$

$= \frac{1}{2}\left[\frac{u^2}{2}\right]_0^{\pi/4}$

$= \frac{1}{4}\left[\frac{\pi^2}{16} - 0\right]$

$= \frac{\pi^2}{64}$ Ans.

Complementary function (C.F.) $= e^{-x/2}\left[c_1 \cos \frac{x\sqrt{3}}{2} + c_2 \sin \frac{x\sqrt{3}}{2}\right]$

$$f(D) = D^2 + D + 1 \text{ and } f(-1) \neq 0$$

Particular integral (P.I.) $= \frac{1}{D^2+D+1} e^{-x}$

$$= \frac{1}{(-1)^2 + (-1) + 1} e^{-x}$$

$$= e^{-x}$$

Hence, solution of the given equation is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = e^{-x/2}\left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right] + e^{-x} \qquad \textbf{Ans.}$$

Example 7.31. Solve the following differential equations :

(i) $(D^2 - 3D + 2)y = e^x$ (ii) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-3x}$

Sol. (i) $(D^2 - 3D + 2)y = e^x$

Auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

Complementary function (C.F.) $= c_1 e^x + c_2 e^{2x}$

Here, $f(D) = D^2 - 3D + 2$ and $f(1) = 0$, where $a = 1$

Particular integral (P.I.) $= \frac{1}{D^2 - 3D + 2} e^x$

$$= \frac{1}{(D-1)(D-2)} e^x$$

We know that, P.I. $= \frac{1}{(D-a)^n \phi(D)} e^{ax} = \frac{x^n}{n!\,\phi(a)} e^{ax}, \phi(a) \neq 0$

Here, $(D-a) = D - 1 \Rightarrow a = 1, n = 1$

and $\phi(D) = D - 2 \Rightarrow \phi(1) = 1 - 2 = -1 \neq 0$

Now P.I. $= \frac{1}{(D-1)(D-2)} e^x$

$$= \frac{x}{1!\,\phi(1)} e^{ax} = \frac{x}{1!}(-1)\, e^x$$

$$\text{P.I.} = -xe^x$$

Hence, the general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = c_1 e^x + c_2 e^{2x} - xe^x \qquad \textbf{Ans.}$$

(ii) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-3x}$

or $(D^2 + 4D + 3)y = e^{-3x}$.

Auxiliary equation is

$$m^2 + 4m + 3 = 0$$

$$(m+1)(m+3) = 0$$

$$m = -1, -3$$

Complementary function (C.F.) $= c_1 e^{-x} + c_2 e^{-3x}$

$$f(D) = D^2 + 4D + 3 = (D+1)(D+3) \text{ and } a = -3$$

$$\therefore \quad f(a) = f(-3) = (-3+1)(-3+3) = 0$$

Particular integral (P.I.) $= \frac{1}{D^2 + 4D + 3} e^{-3x}$

$$= \frac{1}{(D+1)(D+3)} e^{-3x}$$

$$= \frac{1}{D+3} \cdot \frac{1}{-3+1} e^{-3x}$$

$$= -\frac{1}{2} \frac{1}{(D+3)} e^{-3x}$$

$$\text{P.I.} = -\frac{1}{2} \frac{xe^{-3x}}{1!} = -\frac{xe^{-3x}}{2}$$

Complete solution of given equation is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = c_1 e^{-x} + c_2 e^{-3x} - \frac{xe^{-3x}}{2} \qquad \textbf{Ans.}$$

EXERCISE 7.11

Solve the following differential equations :

1. (i) $\frac{d^2y}{dx^2} + y = e^{2x}$ (ii) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{2x}$
2. (i) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$ (ii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2e^x$
3. (i) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 7y = e^x + e^{-x}$ (ii) $\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + (p^2 + q^2)y = e^{ax}$
4. (i) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$ (ii) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$

CHAPTER 8

LAPLACE TRANSFORMATION

8.1 INTRODUCTION

The Laplace transformation is an important operational method for solving linear differential equations, ordinary differential equations as well as partial. The Laplace transformation reduces an ordinary differential equation into an algebraic equation. Furthermore, it takes care of the initial conditions thus avoiding the need of first determining the general solution and then of obtaining the particular solution from it.

8.2 DEFINITION

Let $F(t)$ be a function of t defined for all $t \geq 0$. The Laplace transformation of $F(t)$, denoted by $L\{F(t)\}$ and defined by

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$$

provided that the integral exists and s is a parameter which may be real or complex. It indicates that $f(s)$ is a result of certain operation performed on the given function $F(t)$. In fact $f(s)$ is the product with respect to t from 0 to ∞.

It may be seen that

$$\int_0^\infty e^{-st} F(t)\, dt = \lim_{n \to \infty} \int_0^n e^{-st} F(t)\, dt$$

and the limit may exist only for some specific values of s. The Laplace transform of $F(t)$ is said to exist only for these values of s.

8.3 LAPLACE TRANSFORM OF ELEMENTARY FUNCTIONS

Using the definition, we find the Laplace transformation of some simple functions :

(i) $$L(1) = \frac{1}{s}$$

$\because$ $$L(1) = \int_0^\infty 1.e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty$$

$$= -\frac{1}{s}[0-1] = \frac{1}{s}$$

Hence $$L(1) = \frac{1}{s}$$

(ii) $$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } s>0 \text{ and } n>-1$$

$\because$ $$L\{t^n\} = \int_0^\infty e^{-st} t^n\, dt = \int_0^\infty e^{-x} \cdot \left(\frac{x}{s}\right)^n \frac{dx}{s}, \; st = x$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{n+1-1}\, dx = \frac{\Gamma(n+1)}{s^{n+1}}$$

If $s > 0$ and $n + 1 > 0$

If n be a positive integer, then $\Gamma(n+1) = n!$

Therefore $$L\{t^n\} = \frac{n!}{s^{n+1}}, \text{ if } s>0 \text{ and } n \text{ be a positive integer}$$

(iii) $$L\{e^{at}\} = \frac{1}{s-a}, \text{ if } s>a$$

$\because$ $$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^\infty$$

$$= -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

$$= \frac{-1}{(s-a)}(0-1) = \frac{1}{s-a}$$

$$= \frac{1}{s-a}, \; s>a$$

(iv) $$L\{\cos at\} = \frac{s}{s^2+a^2}, \; L\{\sin at\} = \frac{a}{s^2+a^2}$$

We know that $$L\{e^{at}\} = \frac{1}{s-a}, \text{ if } s>0$$

$\therefore$ $$L\{e^{iat}\} = \frac{1}{s-ia} = \frac{s+ia}{(s-ia)(s+ia)}$$

$$= \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

or $$L\{\cos at + i \sin at\} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

$[\because e^{i\theta} = \cos\theta + i\sin\theta]$

or $$L\{\cos at\} + iL\{\sin at\} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$